Framhaldsskólinn í Vestmannaeyjum

Prófgrein: **STÆ6036**

Próftími: 90 mínútur

klukkan: 8:00-9:30

Dags.

Kennari:

Nafn:	....
Kennitala:	

Hjálpargögn og leiðbeiningar:

Skriffæri, reiknivél ásamt jöfnublaði sem fylgir

Prófið er á 5 tölusettum síðum, athugaðu að ekkert vanti í þitt eintak.
Lesið spurningarnar vel og **svarið því sem spurt er um.**
Sýnið útreikninga þegar það á við

Gangi ykkur vel og gleðilegt sumar

1.(18%) Gefnar eru tvinntölurnar $u = 3 - i$ og $v = 4 + 5i$. Reiknið

a) u-$2v$

b) $u^2 \cdot v$

b) $\frac{u}{v}$

2. (12%) Látum $z = -1 - 3i$ vera tvinntölu. Reiknaðu

a) $\frac{1}{z}$

b) $z \cdot \left(\frac{1}{z}\right)$

3. (7%) Ritaðu tvinntöluna $z = 5 - 12i$ í pólhnitum.

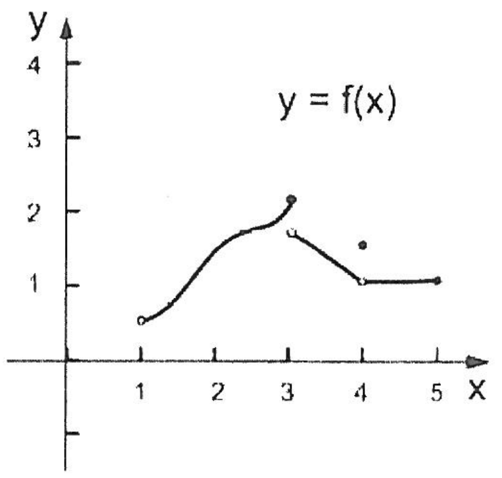
4.(10%) Leystu difffurjöfnuna $4y\frac{dy}{dx} = x$

5. (8%) Leystu diffurjöfnuna $y'' + y' - 6y = 0$

6. (11%) Sýndu að $y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$ sé lausn á diffurjöfnunni $y'' - 2y' + 5y = 0$

7. (14%) Finnið 4. Stigs Taylor margliðu fallsins $f(x) = \cos(x) - e^x$ með miðju $x_0 = 0$.

8. (20%)



Mynd 1: Graf fallsins $f :]1,5] \longrightarrow \mathbb{R}$.

Fyllið út eftirfarandi töflu fyrir fallið sem sést á myndinn hér að ofan.

Merkið **x í þá reiti sem við á** og athugið að möguleiki getur verið að fylla í alla reitina eða engann.

Athugið einnig að í hverju línu þá dragast röng x frá réttum x um- þið fáið semsagt ekki allt rétt þó að þið merkið í alla reitina.

Fallið f í x=	1	2	3	4	5
Hefur markgildi frá vinstri					
Hefur markgildi frá hægri					
Hefur markgildi					
Er samfellt frá vinstri					
Er samfellt frá hægri					
Er samfellt					
Er ósamfellt					

Bónusspurning (10%) Þið hafið val um þessa spurningu og ef þið ákveðið að svara henni gildir það einungis til hækkunar.

Hvers vegna brotnar spaghetti ekki í tvennt þegar ég brýt það?

Hvernig brotnar spaghetti og hvaða lögmál eru að verki?

DIFFERENTIATION RULES

$\frac{d}{dx}\Big(f(x)+g(x)\Big) = f'(x)+g'(x)$

$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$

$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x)+f(x)g'(x)$

$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{\big(f(x)\big)^2}$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{\big(g(x)\big)^2}$

$\frac{d}{dx}f\Big(g(x)\Big) = f'\Big(g(x)\Big)g'(x)$

ELEMENTARY DERIVATIVES

$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}x^r = rx^{r-1}$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}a^x = a^x \ln a \quad (a>0)$

$\frac{d}{dx}\ln x = \frac{1}{x} \quad (x>0)$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx}|x| = \operatorname{sgn} x = \frac{x}{|x|}$

TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1$

$\sec^2 x = 1 + \tan^2 x$

$\csc^2 x = 1 + \cot^2 x$

$\sin(-x) = -\sin x$

$\sin(\pi - x) = \sin x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$\cos(-x) = \cos x$

$\cos(\pi - x) = -\cos x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

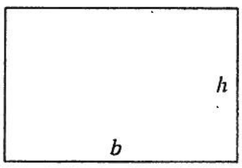
QUADRATIC FORMULA

If $Ax^2 + Bx + C = 0$, then $x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$.

GEOMETRIC FORMULAS

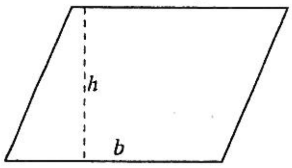
A = area, b = base, h = height, C = circumference, V = volume, S = surface area

Rectangle



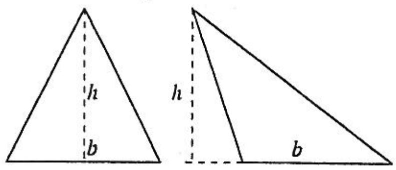
$A = bh$

Parallelogram



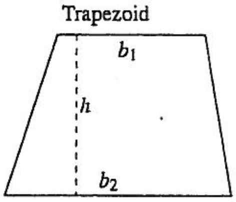
$A = bh$

Triangles



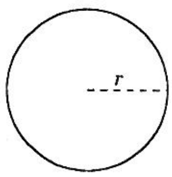
$A = \frac{1}{2}bh$

Trapezoid



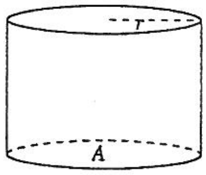
$A = \frac{1}{2}(b_1+b_2)h$

Circle



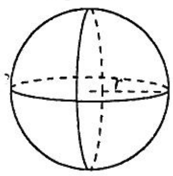
$A = \pi r^2$, $C = 2\pi r$

Circular Cylinder



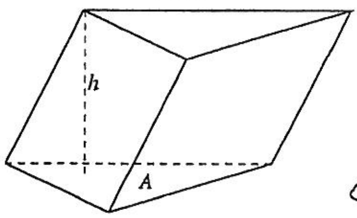
$V = Ah = \pi r^2 h$, $S = Ch = 2\pi rh$ (cylindrical wall)

Sphere



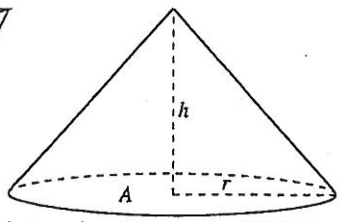
$V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$

Prism



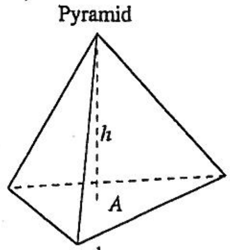
$V = Ah$

Circular Cone



$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h$, $S = \pi r\sqrt{r^2+h^2}$ (conical wall)

Pyramid



$V = \frac{1}{3}Ah$

INTEGRALS INVOLVING $\sqrt{x^2 \pm a^2}$ $(a > 0)$

(If $\sqrt{x^2 - a^2}$, assume $x > a > 0$.)

$$\int \sqrt{x^2 \pm a^2}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{\sqrt{x^2 + a^2}}{x}\, dx = \sqrt{x^2 + a^2} - a \ln\left|\frac{a + \sqrt{x^2 + a^2}}{x}\right| + C$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a \tan^{-1}\frac{\sqrt{x^2 - a^2}}{a} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\, dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{\sqrt{x^2 \pm a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 \pm a^2}}{x} + \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x} + C$$

$$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}} + C$$

$$\int (x^2 \pm a^2)^{3/2}\, dx = \frac{x}{8}(2x^2 \pm 5a^2)\sqrt{x^2 \pm a^2} + \frac{3a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

INTEGRALS INVOLVING $\sqrt{a^2 - x^2}$ $(a > 0, |x| < a)$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + C$$

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$$

$$\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$$

$$\int \frac{\sqrt{a^2 - x^2}}{x^2}\, dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + C$$

$$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}} + C$$

$$\int (a^2 - x^2)^{3/2}\, dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1}\frac{x}{a} + C$$

INTEGRALS OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\int \sin^{-1}x\, dx = x \sin^{-1}x + \sqrt{1 - x^2} + C$$

$$\int \tan^{-1}x\, dx = x \tan^{-1}x - \frac{1}{2} \ln(1 + x^2) + C$$

$$\int \sec^{-1}x\, dx = x \sec^{-1}x - \ln|x + \sqrt{x^2 - 1}| + C \quad (x > 1)$$

$$\int x \sin^{-1}x\, dx = \frac{1}{4}(2x^2 - 1)\sin^{-1}x + \frac{x}{4}\sqrt{1 - x^2} + C$$

$$\int x \tan^{-1}x\, dx = \frac{1}{2}(x^2 + 1)\tan^{-1}x - \frac{x}{2} + C$$

$$\int x \sec^{-1}x\, dx = \frac{x^2}{2} \sec^{-1}x - \frac{1}{2}\sqrt{x^2 - 1} + C \quad (x > 1)$$

$$\int x^n \sin^{-1}x\, dx = \frac{x^{n+1}}{n+1}\sin^{-1}x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1 - x^2}}\, dx + C \text{ if } n \neq -1$$

$$\int x^n \tan^{-1}x\, dx = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1} \int \frac{x^{n+1}}{1 + x^2}\, dx + C \text{ if } n \neq -1$$

$$\int x^n \sec^{-1}x\, dx = \frac{x^{n+1}}{n+1}\sec^{-1}x - \frac{1}{n+1} \int \frac{x^n}{\sqrt{x^2 - 1}}\, dx + C \quad (n \neq -1, x$$

EXPONENTIAL AND LOGARITHMIC INTEGRALS

$$\int xe^x\, dx = (x - 1)e^x + C$$

$$\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx$$

$$\int \ln x\, dx = x \ln x - x + C$$

$$\int x^n \ln x\, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C, \quad (n \neq -1)$$

$$\int x^n (\ln x)^m\, dx = \frac{x^{n+1}}{n+1}(\ln x)^m - \frac{m}{n+1} \int x^n (\ln x)^{m-1}\, dx \quad (n \neq -1)$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$$

INTEGRALS OF HYPERBOLIC FUNCTIONS

$$\int \sinh x\, dx = \cosh x + C$$

$$\int \cosh x\, dx = \sinh x + C$$

$$\int \tanh x\, dx = \ln(\cosh x) + C$$

$$\int \coth x\, dx = \ln|\sinh x| + C$$

$$\int \operatorname{sech} x\, dx = 2\tan^{-1}(e^x) + C$$

$$\int \operatorname{csch} x\, dx = \ln\left|\tanh\frac{x}{2}\right| + C$$

$$\int \sinh^2 x\, dx = \frac{1}{4} \sinh 2x - \frac{x}{2} + C$$

$$\int \cosh^2 x\, dx = \frac{1}{4} \sinh 2x + \frac{x}{2} + C$$

$$\int \tanh^2 x\, dx = x - \tanh x + C$$

$$\int \coth^2 x\, dx = x - \coth x + C$$

$$\int \operatorname{sech}^2 x\, dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x\, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$$

Formúlublað í STÆ 403

Diffurreglur

$f(x)$	$f'(x)$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
x^a	$a \cdot x^{a-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$ $1+\tan^2(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
a^x	$a^x \cdot \ln(a)$
$\log_a(x)$	$\frac{1}{x \cdot \ln(a)}$

$j(x)$	$j'(x)$
$f(x)+g(x)$	$f'(x)+g'(x)$
$f(x)-g(x)$	$f'(x)-g'(x)$
$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
$f \circ g(x)$	$f'(g(x)) \cdot g'(x)$
$(f^{-1})'(y)$	$\frac{1}{f'(x)}$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

Snertill:
$$y - f(x_0) = f'(x_0)(x - x_0)$$

Hornaföll:

$$\sin^2(v) + \cos^2(v) = 1$$

$$\tan^2(v) + 1 = \frac{1}{\cos^2(v)}$$

$$\sin(2v) = 2\sin(v) \cdot \cos(v)$$

$$\cos(2v) = \cos^2(v) - \sin^2(v)$$

$$\cos(2v) = 1 - 2\sin^2(v)$$

$$\cos(2v) = 2\cos^2(v) - 1$$

$$\tan(2v) = \frac{2\tan(v)}{1-\tan^2(v)}$$

Lograreglur

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log(a^n) = n \cdot \log(a)$$

$$\sin(u) - \sin(v) = 2\cos\left(\frac{u+v}{2}\right) \cdot \sin\left(\frac{u-v}{2}\right)$$

$$\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right) \cdot \cos\left(\frac{u-v}{2}\right)$$

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right) \cdot \sin\left(\frac{u-v}{2}\right)$$

$$\cos(u) + \cos(v) = 2\cos\left(\frac{u+v}{2}\right) \cdot \cos\left(\frac{u-v}{2}\right)$$

Helstu aðgerðirnar eru skilgreindar á eftirfarandi hátt, þar sem $z = x_1 + y_1 i$ og $w = x_2 + y_2 i$:

- Samsemd: $z = w$ þ.þ.a.a. $x_1 = x_2$ og $y_1 = y_2$
- Samlagning: $z + w = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$
- Margföldun:
  $zw = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$
- Deiling: $\frac{z}{w} = \frac{z\bar{w}}{|w|^2} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{x_2^2 + y_2^2}$
- Margföldunarandhverfa: Ef $zw = 1$ eru tvinntölurnar sagðar (margföldunar)andhverfa hvor annarrar. Þá er:

$z = \frac{\bar{w}}{|w|^2}$ eða $w = \frac{\bar{z}}{|z|^2}$

Ef $z = x + yi$ er talan $\bar{z} = x - yi$ sögð vera samoka henni. Samoka tvinntölur hafa ýmsa áhugaverða eiginleika:

$$z + \bar{z} = 2x$$
$$z - \bar{z} = 2yi$$
$$z\bar{z} = |z|^2$$
$$\frac{z}{\bar{z}} = \frac{z^2}{|z|^2}$$

] Hlutar tvinntölu

Til þess að fá uppgefinn eingöngu raunhluta tvinntölunnar $z = x + yi$ er notað fallið $\Re(z)$ og er það skilgreint svo:

$$\Re(z) = x = \frac{1}{2}(z + \bar{z})$$

Fyrir þverhlutann er fallið $\Im(z)$ notað, en skilgreining þess er:

$$\Im(z) = y = \frac{1}{2i}(z - \bar{z})$$

Í stað $\Re(z)$ og $\Im(z)$ er oft ritað Re(z) og Im(z). Báðir rithættirnir eru jafngildir þar sem Re stendur fyrir *real* og á við raunhlutann og Im stendur fyrir *imaginary* og á við þverhlutann. Sérstaka athygli skal vekja á því að $\Im(z)$ gefur **rauntöluna** y, ekki þvertöluna yi.

Veldi

Hægt er að rita tvinntölur sem veldi af e, þ.e. e^z þar sem að z er tvinntala.

$$z = x + yi$$
$$e^z = e^{x+yi} = e^x \cdot e^{yi}$$

Raunhluti tvinntölunnar er þá samsvarandi fallinu e^x (náttúrulega vísisfallið), en um þverhlutann gildir að

$$e^{yi} = cos(y) + isin(y)$$

Þar sem að allar rétthyrndar tvinntölur má líka rita í pólhnitum, þannig að

$$z = r \cdot (\cos(\theta) + i\sin(\theta))$$

má segja að:

$$z = x + iy = (r,\theta) = re^{i\theta}$$

Margföldun tvinntalna í þessu formi er þannig að:

$$z_1 \cdot z_2 = r_1 e^{i\phi_1} \cdot r_2 e^{i\phi_2} = r_1 r_2 \cdot e^{i\phi_1 + i\phi_2} = r_1 r_2 \cdot e^{i(\phi_1 + \phi_2)}$$

Hægt er að leiða reglu Eulers út frá þessu:

$$e^{i\phi} = \cos(\phi) + i\sin(\phi) \wedge e^{-i\phi} = \cos(\phi) - i\sin(\phi)$$
$$\cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2} \wedge \sin(\phi) = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$